

An elliptic theory of indicial weights and applications to non-linear geometry problems

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Abstract

Given an elliptic operator P on a non-compact manifold (with proper asymptotic conditions), there is a discrete set of numbers called indicial roots. It's known that P is Fredholm between weighted Sobolev spaces if and only if the weight is not indicial. We show that an elliptic theory exists even when the weight is indicial. We also discuss some simple applications to Yang-Mills theory and minimal surfaces.

1 Introduction

1.1 The theory

The elliptic theories based on weighted Sobolev (Schauder) spaces usually concern a discrete set of real numbers. If a number is in the set, we say that it is indicial (or is an indicial root). A classical fact says that on a non-compact complete manifold, an elliptic operator (with proper asymptotic conditions) is Fredholm between weighted Sobolev spaces if and only if the weight is not indicial. For earlier pioneering work, please see [10], [11], and [14]. For more recent work, please see [12].

Following elementary ideas, we show that there is an elliptic theory even if the weight is indicial: first, we add polynomial weights {compare (6) to [11, (1.3)]} to refine the space; second, we consider graph norms with respect to the model operator [see (2)].

In this note we only consider first and second-order operators modelled on the following.

Definition 1.1. Let Y be a $(n-1)$ -dimensional Riemannian manifold without boundary (which does not have to be connected). Let E, F be smooth vector-bundles over Y equipped with smooth Hermitian metrics. Given arbitrary bundle isomorphisms $\sigma_1 : E \rightarrow F$, $\sigma_2 : E \rightarrow E$, we say that an operator P^0 is TID (translation-invariant and diagonal) if

$$P^0 = \sigma_1(-B_{P^0} - a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2})\sigma_2 \text{ and the following holds.} \quad (1)$$

- $a_2 = 0$ or 1 . $a_1 = -1$ when $a_2 = 0$ (always achievable by normalization).
- When $a_2 = 0$, B_{P^0} is a first-order self-adjoint elliptic differential operator $C^\infty(Y, E) \rightarrow C^\infty(Y, E)$. When $a_2 = 1$, B_{P^0} is second-order, simple, elliptic, and self-adjoint $C^\infty(Y, E) \rightarrow C^\infty(Y, E)$ (see Definition 2.1).

Remark 1.2. For any TID operator P^0 , $\text{Spec} B_{P^0}$ is real and discrete. Moreover, there is a complete eigen-basis of B_{P^0} .

Definition 1.3. Let P^0 be TID, and (β, Λ) be a pair of real numbers such that $\Lambda \in \text{Spec}(B_{P^0})$. When P^0 is first-order, we say that (β, Λ) is P^0 -indicial if $\beta = \Lambda$. When P^0 is second-order, we say that (β, Λ) is P^0 -indicial if

1. $\beta \neq \frac{a_1}{2}$ and $\Lambda - \beta^2 + a_1\beta = 0$, or if
2. $\Lambda \leq -\frac{a_1^2}{4}$ and $\beta = \frac{a_1}{2}$.

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In the second case above, we say that (β, Λ) is P^0 -super indicial. We say that β is P^0 -indicial (super indicial) if there is a $\Lambda \in \text{Spec} B_{P^0}$ such that (β, Λ) is P^0 -indicial (super indicial). This is consistent with the " \mathfrak{D}_A " in [11, page 417], translated to our setting.

Let N be a complete Riemannian manifold with finite many cylindrical ends, we consider asymptotically TID operators $P : C^\infty(N, E) \rightarrow C^\infty(N, F)$. This class should include most of the Dirac and Laplace-type operators in geometry. In the setting as Theorem 1.4,

$$\begin{aligned} P : \widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}(N, E) &\longrightarrow W_{-\beta, \gamma, b}^{k, p}(N, F) & (\square|_{-\beta, \gamma, b}^{\text{Sobolev}, p}) \text{ (see Definition 2.3, 2.7)} \\ P : \widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}(N, E) &\longrightarrow C_{-\beta, \gamma, b}^{k, \alpha}(N, F) & (\square|_{-\beta, \gamma, b}^{\text{Schauder}}) \text{ for the norms).} \end{aligned} \quad (2)$$

are bounded operators. Moreover, when β is not indicial and $\gamma = 0$, as subspaces of L_{loc}^2 ,

$$\begin{aligned} \widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}(N, E) &= W_{-\beta}^{k+m_0, p}(N, E), \quad \widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}(N, E) = C_{-\beta}^{k+m_0, \alpha}(N, E), \\ W_{-\beta, \gamma, b}^{k, p}(N, F) &= W_{-\beta}^{k, p}(N, F), \quad C_{-\beta, \gamma, b}^{k, \alpha}(N, F) = C_{-\beta}^{k, \alpha}(N, F). \end{aligned} \quad (3)$$

Thus our theory generalizes the one in [11] {for first and second-order operators, c.f [11, (1.3)]}. Assuming the weights are the same on the ends, our main result states as follows.

Theorem 1.4. *Suppose P is a β -ATID elliptic operator (see Definition 2.8), and β is not P^0 -super indicial. Then for any $0 \leq k \leq k_0 - 2$, $\alpha \in (0, 1)$, $p \geq 2$,*

- $(\square|_{-\beta, \gamma, b}^{\text{Sobolev}, p})$ is Fredholm if $b \neq 1 - \frac{1}{p}$ or β is not P^0 -indicial;
- $(\square|_{-\beta, \gamma, b}^{\text{Schauder}})$ is Fredholm if $b \neq 1$ or β is not P^0 -indicial.

Remark 1.5. The super-indicial roots are essentially different from the ordinary ones. Fortunately, they don't exist for first-order operators, and they barely appear on second-order operators. For example, any super-indicial root in the setting of Corollary 1.9 must be positive, but we only need β to be non-positive therein.

Remark 1.6. Theorem 5.2 and Proposition 4.5 give reasonably general index formulas for first-order operators (see Remark 5.3). As a by-product, we prove an obvious identity (Proposition 5.4) on the eta-invariant defined in [3]. It can also be proved by the Fredholm theory in [3]. However, the author is not able to find Proposition 5.4 in the literature.

Remark 1.7. Though the indicial roots do not prevent Fredholmness, the index still changes when β goes across any of them (c.f. [11, Last 5 lines in Page 433]).

Our theory still works when the weights are not the same on the ends (see Theorem 2.11).

Computations indicate that the our local inverses (Theorem 3.2) are different from those of Lockhart-McOwen [11, (2.3)] (by Fourier-transform in the t -direction). When $k < 0$, our local inverses do not work for the $W^{k, p}$ ($C^{k, \alpha}$) theories.

Remark 1.8. Assuming that P is translation invariant on each end, Theorem 1.4, 2.11 are still true with "if" replaced by "if and only if" (see the proof in the Appendix). However, only assuming β -ATID, when $b = 1$, we don't know whether $(\square|_{-\beta, \gamma, b}^{\text{Schauder}})$ is not Fredholm. The same doubt applies to the Sobolev theory.

By simple conformal changes as in [11, Section 9], Theorem 1.4 is equivalent to a theory in the conic setting (and hopefully the asymptotic conic setting). Please also see [15] and the discussion above Lemma 7.2.

Under stronger asymptotic conditions than Definition 2.8, we have a theory for super-indicial roots (on second-order operators), and a theory for powers of Laplace-type operators i.e. Δ^m , ($m \geq 2$). The higher-order operators include (linearisation of) the extremal metric operator in [4], and the conformal co-variant operators in [5]. We will address these in the future when geometric motivation arises.

Amrouch-Girault-Giroire [2] also use Sobolev-spaces with polynomial weights to study Laplace equations on domains. It's possible that our theory is essentially similar to theirs.

1.2 Simple applications

Geometric objects with isolated conic singularities usually converge to their tangent cones polynomially (see [16]). Let r be the distance to the singular point, and $t = -\log r$ be the cylindrical coordinate. Our work implies a general phenomenon: the rate of convergence to the tangent cone is either exponential or not faster than $\frac{1}{t}$ (or $\frac{1}{-\log r}$).

We first do minimal sub-manifolds. In the cylindrical setting, we say that a minimal graph sub-manifold is asymptotic to a cone at a certain rate, if the section "u" in (63) converges to 0 at the rate (see Definition 6.1).

Corollary 1.9. *Suppose Σ is a n -dimensional closed minimal sub-manifold in \mathbb{S}^N , $n \geq 1$. Let \underline{Q} be the negative number in Definition 3.1 with respect to the L_Σ in (63). Then there is a δ_0 depending on Σ with the following property.*

Suppose $\widehat{\Sigma}$ is a (locally defined) embedded minimal sub-manifold in \mathbb{R}^{N+1} with isolated cone singularity at O . Suppose $\widehat{\Sigma}$ is a graph over $\text{Cone}(\Sigma)$, and in the cylindrical setting, it converges to $\text{Cone}(\Sigma)$ at least at the rate $\frac{\delta_0}{t}$ (see Definition 6.1). Then $\widehat{\Sigma}$ converges to $\text{Cone}(\Sigma)$ exponentially at the rate $O(e^{-|\underline{Q}|t})$.

Remark 1.10. By Definition 2.8 and Remark 2.9, we can not make δ_0 small by scaling. Adam-Simon [1] showed that there are singular minimal sub-manifolds converging to a cone at a rate comparable to $(-\log r)^{-1}$. This suggests that in general, the assumption on the rate in Corollary 1.9 can not be weakened.

Similar results hold for Yang-Mills connections as well.

Corollary 1.11. *Let $n \geq 5$. Suppose g is a smooth metric on $B_O^n(R)$ and $g(O) = g_E$ (the Euclidean metric). Suppose A_O is a $U(m)$ or $SO(m)$ Yang-Mills connection on (the unit round) \mathbb{S}^{n-1} . Let \underline{Q} be the negative number in Definition 3.1 with respect to the B in (60). Then there is a $\delta_0 > 0$ depending on A_O with the following properties.*

Suppose A is a smooth Yang-Mills connection on $B_O(R) \setminus O$. In the cylindrical setting as Section 6.1, suppose A converges to $\text{Cone}(A_O)$ at least at the rate $\frac{\delta_0}{t}$ (see Definition 6.1).

I : Suppose A is in Coulomb gauge relative to A_O (with respect to g or the Euclidean metric). Then A converges to $\text{Cone}(A_O)$ exponentially at the following rate.

$$\begin{cases} O(e^{-|\underline{Q}|t}) & \text{when } |\underline{Q}| < 1, \\ O(e^{-t}) & \text{when } |\underline{Q}| > 1, \\ O(te^{-|\underline{Q}|t}) & \text{when } |\underline{Q}| = 1. \end{cases} \quad (4)$$

II : When A_O is irreducible, there exists a gauge s such that $s(A)$ converges to $\text{Cone}(A_O)$ exponentially as (4).

Remark 1.12. By (59), when $n = 4$, the weight 0 is super-indicial unless B is positively definite.

Organization of this note: the norms can be found in Section 2. We construct the local inverses in Section 3. In Section 4, we study regularity of harmonic sections, and complete the proof for Theorem 1.4. We give the index formula (for first-order operators) and study the eta-invariant in Section 5. We prove Corollary 1.9, 1.11 in Section 6.

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2 Preparation

Definition 2.1. In the setting of Definition 1.1, we say that an operator $H : C^\infty(Y, E) \rightarrow C^\infty(Y, E)$ is admissible, if there is a linear first-order differential operator H_0 , and sections $\widehat{\Gamma}_k, \Gamma_k \in C^\infty(Y, E)$ such that

$$H\xi = H_0\xi + \sum_{k=1}^{k_0} \widehat{\Gamma}_k \int_Y \langle \xi, \Gamma_k \rangle dV. \quad (5)$$

We say that a second-order operator $B : C^\infty(Y, E) \rightarrow C^\infty(Y, E)$ is simple, if there is a smooth connection A_0 on E and a smooth metric \widehat{g} on Y (which does not need to be the given one), such that $B - \nabla_{A_0}^{\star \widehat{g}} \nabla_{A_0}$ is admissible.

Definition 2.2. (Strips) Let $kS_m = (m - k, m + k)$, and $\{\xi_m\}_{m=3}^\infty$ be a partition of unity of Cyl_1 subordinate to the cover $\{2S_m\}_{m=3}^\infty$ i.e. ξ_m is supported in $2S_m$ and is $\equiv 1$ in S_m .

Definition 2.3. Let Cyl_{t_0} denote $Y \times (t_0, \infty)$, $t_0 \geq 0.1$, we define the $L_{-\beta, b}^2[Cyl_{t_0}]$ - space of sections to the underlying bundle by the norm

$$|\xi|_{L_{-\beta, b}^p(Cyl_{t_0})} \triangleq |e^{-\beta t} t^b \xi|_{L^p(Cyl_{t_0})} = \left(\int_{Cyl_{t_0}} |e^{-\beta t} t^b \xi|^p \right)^{\frac{1}{p}}. \quad (6)$$

We define $|\xi|_{W_{-\beta, b}^{k, p}(Cyl_{t_0})} \triangleq \sum_{j=0}^k |\nabla_{A_0}^j \xi|_{L_{-\beta, b}^p(Cyl_{t_0})}$. For the Schauder theory, we define $|\xi|_{C_{-\beta, b}^{k, \alpha}(\overline{Cyl}_{t_0})} \triangleq \sup_{m \geq t_0+1} m^b e^{-\beta m} |\xi|_{C^{k, \alpha}(S_m)}$.

Let $\xi^{\perp \beta}$ denote the projection of ξ onto $\text{Ker}\{B_{P^0} - \beta Id\}$ (for all t), and $\xi^{\perp \beta} = \xi - \xi^{\parallel \beta}$ be the perpendicular vector. When P^0 is first-order, we define

$$\begin{aligned} |\sigma_2^{-1} \xi|_{\widehat{W}_{-\beta, \gamma, b-1}^{k, p, P^0}(Cyl_{t_0})} &\triangleq |\xi^{\perp \beta}|_{W_{-\beta, \gamma}^{k, p}(Cyl_{t_0})} + |\xi^{\parallel \beta}|_{W_{-\beta, b-1}^{k, p}(Cyl_{t_0})} + \left| \frac{\partial \xi^{\parallel \beta}}{\partial t} - \beta \xi^{\parallel \beta} \right|_{W_{-\beta, b}^{k-1, p}(Cyl_{t_0})} \\ |\sigma_2^{-1} \xi|_{W_{-\beta, \gamma, b}^{k, p, P^0}(Cyl_{t_0})} &\triangleq |\xi^{\perp \beta}|_{W_{-\beta, \gamma}^{k, p}(Cyl_{t_0})} + |\xi^{\parallel \beta}|_{W_{-\beta, b}^{k, p}(Cyl_{t_0})} \\ |\sigma_2^{-1} \xi|_{\widehat{C}_{-\beta, \gamma, b-1}^{k, \alpha, P^0}(\overline{Cyl}_{t_0})} &\triangleq |\xi^{\perp \beta}|_{C_{-\beta, \gamma}^{k, \alpha}(\overline{Cyl}_{t_0})} + |\xi^{\parallel \beta}|_{C_{-\beta, b-1}^{k, \alpha}(\overline{Cyl}_{t_0})} + \left| \frac{\partial \xi^{\parallel \beta}}{\partial t} - \beta \xi^{\parallel \beta} \right|_{C_{-\beta, b}^{k-1, \alpha}(\overline{Cyl}_{t_0})} \\ |\sigma_2^{-1} \xi|_{C_{-\beta, \gamma, b}^{k, \alpha, P^0}(\overline{Cyl}_{t_0})} &\triangleq |\xi^{\perp \beta}|_{C_{-\beta, \gamma}^{k, \alpha}(\overline{Cyl}_{t_0})} + |\xi^{\parallel \beta}|_{C_{-\beta, b}^{k, \alpha}(\overline{Cyl}_{t_0})}. \end{aligned}$$

When P^0 is second-order elliptic, let $\Lambda_\beta = \beta^2 - a_1 \beta$, we define

$$\begin{aligned} |\sigma_2^{-1} \xi|_{\widehat{W}_{-\beta, \gamma, b-1}^{k, p, P^0}(Cyl_{t_0})} &\triangleq |\xi^{\perp \Lambda_\beta}|_{W_{-\beta, \gamma}^{k, p}(Cyl_{t_0})} + |\xi^{\parallel \Lambda_\beta}|_{W_{-\beta, b-1}^{k, p}(Cyl_{t_0})} + \left| \frac{\partial \xi^{\parallel \Lambda_\beta}}{\partial t} - \beta \xi^{\parallel \Lambda_\beta} \right|_{W_{-\beta, b}^{k-1, p}(Cyl_{t_0})} \\ &\quad + \left| \frac{\partial^2 \xi^{\parallel \Lambda_\beta}}{\partial t^2} - \beta^2 \xi^{\parallel \Lambda_\beta} \right|_{W_{-\beta, b}^{k-2, p}(Cyl_{t_0})} \\ |\sigma_2^{-1} \xi|_{W_{-\beta, \gamma, b}^{k, p, P^0}(Cyl_{t_0})} &\triangleq |\xi^{\perp \Lambda_\beta}|_{W_{-\beta, \gamma}^{k, p}(Cyl_{t_0})} + |\xi^{\parallel \Lambda_\beta}|_{W_{-\beta, b}^{k, p}(Cyl_{t_0})} \\ |\sigma_2^{-1} \xi|_{\widehat{C}_{-\beta, \gamma, b-1}^{k, \alpha, P^0}(\overline{Cyl}_{t_0})} &\triangleq |\xi^{\perp \Lambda_\beta}|_{C_{-\beta, \gamma}^{k, \alpha}(\overline{Cyl}_{t_0})} + |\xi^{\parallel \Lambda_\beta}|_{C_{-\beta, b-1}^{k, \alpha}(\overline{Cyl}_{t_0})} + \left| \frac{\partial \xi^{\parallel \Lambda_\beta}}{\partial t} - \beta \xi^{\parallel \Lambda_\beta} \right|_{C_{-\beta, b}^{k-1, \alpha}(\overline{Cyl}_{t_0})} \\ &\quad + \left| \frac{\partial^2 \xi^{\parallel \Lambda_\beta}}{\partial t^2} - \beta^2 \xi^{\parallel \Lambda_\beta} \right|_{C_{-\beta, b}^{k-2, \alpha}(\overline{Cyl}_{t_0})} \\ |\sigma_2^{-1} \xi|_{C_{-\beta, \gamma, b}^{k, \alpha, P^0}(\overline{Cyl}_{t_0})} &\triangleq |\xi^{\perp \Lambda_\beta}|_{C_{-\beta, \gamma}^{k, \alpha}(\overline{Cyl}_{t_0})} + |\xi^{\parallel \Lambda_\beta}|_{C_{-\beta, b}^{k, \alpha}(\overline{Cyl}_{t_0})}. \end{aligned}$$

Remark 2.4. The σ_2 of the adjoint operator L^* in (52) is usually not identity, but it never affects the index or kernel.

For all first and second-order TID-operators, we abuse notation and denote the corresponding operators on the link as B_{P^0} . We need to solve the equations

$$\left(\frac{\partial}{\partial t} - B_{P^0} \right) \sigma_2 \xi = \sigma_1^{-1} h, \quad \left(\frac{\partial^2}{\partial t^2} - a_1 \frac{\partial}{\partial t} - B_{P^0} \right) \sigma_2 \xi = \sigma_1^{-1} h \text{ respectively.} \quad (7)$$

Let $u = (\sigma_2 \xi) e^{-\beta t}$, $f = (\sigma_1^{-1} h) e^{-\beta t}$, (7) become

$$\frac{\partial u}{\partial t} - B_{P_\beta^0} u = f \text{ and } \frac{\partial^2 u}{\partial t^2} - (a_1 - 2\beta) \frac{\partial u}{\partial t} - B_{P_\beta^0} u = f \text{ respectively,} \quad (8)$$

where

$$B_{P_\beta^0} \triangleq \begin{cases} B_{P^0} - \beta Id & \text{when } P^0 \text{ is first-order} \\ B_{P^0} + a_1 \beta - \beta^2 & \text{when } P^0 \text{ is second-order elliptic.} \end{cases} \quad (9)$$

For any $\Lambda \in \text{Spec}(B_{P^0})$ (repeated by multiplicity). Let

$$\lambda [\in \text{Spec}(B_{P_\beta^0})] \triangleq \begin{cases} \Lambda - \beta & \text{when } P^0 \text{ is first-order,} \\ \Lambda + a_1\beta - \beta^2 & \text{when } P^0 \text{ is second-order elliptic.} \end{cases} \quad (10)$$

Let $[\phi_\Lambda, \Lambda \in \text{Spec}(B_{P^0})]$ denote the orthonormal eigen-basis of $L^2[Y, E]$ with respect to B_{P^0} . Abusing notation, we let $\phi_\lambda = \phi_\Lambda$. In terms of the Fourier series $u(f) = \sum_\Lambda u_\lambda \phi_\Lambda$ ($\sum_\Lambda f_\lambda \phi_\Lambda$), (8) is equivalent to the ODE's

$$\frac{du_\lambda}{dt} - \lambda u_\lambda = f_\lambda, \quad \frac{d^2 u_\lambda}{dt^2} - (a_1 - 2\beta) \frac{du_\lambda}{dt} - \lambda u_\lambda = f_\lambda \text{ for all } \lambda \in \text{Spec} B_{P_\beta^0} \text{ respectively.} \quad (11)$$

Remark 2.5. Let $\sigma_2 = Id$. In terms of the Fourier-coefficients, when P^0 is first-order,

$$\begin{aligned} |\xi|_{\widehat{W}_{0,\gamma,b-1}^{1,2,P_\beta^0}}(Cyl_{t_0}) &= \sum_{\lambda \in \text{Spec}(B_{P_\beta^0}), \lambda \neq 0} [(1 + \lambda^2) \int_{t_0}^\infty \xi_\lambda^2 s^{2\gamma} ds + \int_{t_0}^\infty |\frac{d\xi_\lambda}{ds}|^2 s^{2\gamma} ds] \\ &\quad + \sum_{\lambda \in \text{Spec}(B_{P_\beta^0}), \lambda = 0} [\int_{t_0}^\infty \xi_\lambda^2 s^{2b-2} ds + \int_{t_0}^\infty |\frac{d\xi_\lambda}{ds}|^2 s^{2b} ds]. \end{aligned}$$

When P^0 is second-order elliptic, using the usual $W^{2,2}$ -elliptic estimate on strips, we routinely verify the following for any ξ compactly supported in $Cyl_{t_0+\epsilon}$.

$$\begin{aligned} |\xi|_{\widehat{W}_{0,\gamma,b-1}^{2,2,P_\beta^0}}(Cyl_{t_0}) &\leq C(\epsilon) \{ \sum_{\lambda \in \text{Spec}(B_{P_\beta^0}), \lambda \neq 0} \int_{t_0}^\infty [(1 + \lambda^2) \xi_\lambda^2 + (1 + |\lambda|) |\frac{d\xi_\lambda}{dt}|^2 + |\frac{d^2 \xi_\lambda}{dt^2}|^2] t^{2\gamma} dt \\ &\quad + \sum_{\lambda \in \text{Spec}(B_{P_\beta^0}), \lambda = 0} [\int_{t_0}^\infty \xi_\lambda^2 t^{2b-2} ds + \int_{t_0}^\infty (|\frac{d\xi_\lambda}{dt}|^2 + |\frac{d^2 \xi_\lambda}{dt^2}|^2) t^{2b} dt] \} \end{aligned}$$

Remark 2.6. Multiplying by $e^{-\beta t}$ is a linear isomorphism:

$$\widehat{C}_{-\beta,\gamma,b-1}^{k,\alpha,P^0} \longrightarrow \widehat{C}_{0,\gamma,b-1}^{k,\alpha,P_\beta^0}, \quad \widehat{W}_{-\beta,\gamma,b-1}^{k,p,P^0} \longrightarrow \widehat{W}_{0,\gamma,b-1}^{k,p,P_\beta^0}, \quad C_{-\beta,\gamma,b}^{k,\alpha} \longrightarrow C_{0,\gamma,b}^{k,\alpha}, \quad L_{-\beta,\gamma,b}^2 \longrightarrow L_{0,\gamma,b}^2.$$

Definition 2.7. Suppose $\vec{\beta} = (\beta_1, \dots, \beta_{l_0})$, $\vec{\gamma} = (\gamma_1, \dots, \gamma_{l_0})$, $\vec{b} = (b_1, \dots, b_{l_0})$ are vectors of l_0 -entries. Given an *ATID* operator P over a manifold N with l_0 cylindrical ends, we denote the ends by U_j , $j = 1 \dots l_0$. We add the interior U_0 to obtain an open cover of N . Using a partition of unity χ_j , $j = 0 \dots l_0$ subordinate to the cover, we define

$$|\xi|_{\widehat{W}_{-\beta,\gamma,b-1}^{k,p,P}}(N) = |\chi_0 \xi|_{W^{k,2}(U_0)} + \sum_{j=1}^{l_0} |\chi_j \xi|_{\widehat{W}_{-\beta_j,\gamma_j,b_j-1}^{k,p,P^{0,j}}(U_j)}, \quad (12)$$

where $P^{0,j}$ is the limit *TID* operator of P on the j -th end. The same definition as (12) applies to all the other norms in Definition 2.3 (including $\widehat{C}_{-\beta,\gamma,b-1}^{k,\alpha,P}(N)$, $C_{-\beta,\gamma,b}^{k,\alpha,P}(N)$ etc).

When the domain is the whole manifold, we usually hide the N in the norm symbols.

Important Convention: When $\beta_1 = \dots = \beta_{l_0} = \beta$, we denote $\vec{\beta}$ (a vector) as β (number). The same applies to \vec{b} and $\vec{\gamma}$. This makes the notations consistent.

Definition 2.8. Let $\delta_0 > 0$ be small enough with respect to the data in Theorem 3.2 except t_0 , such that the Neumann-Series in Lemma 4.4 and Theorem 3.3 converge as desired.

Let $k_0 \geq 10$, $|\cdot|_{C^k}(t, y) \triangleq \sum_{0 \leq i+j \leq k} |\frac{\partial}{\partial t^i} \nabla^j \cdot|(t, y)$. We say that P satisfies the $\mathbb{S}_\beta(l_1, l_2)|_{Cyl_{t_0}}$ -condition, if the following holds for any $k \leq k_0$, $t \geq t_0 + 1$, $\alpha \in [0, 1)$, ξ , and a δ_0 small enough with respect to the data in Theorem 3.2.

$$\begin{aligned} t^{l_1} |(P - P^0) \xi^{\perp \beta}|_{C^{k,\alpha}(\overline{S}_t)} &\leq \delta_0 |\xi^{\perp \beta}|_{C^{k+m_0,\alpha}(\overline{S}_t)}, \quad t^{l_1} |(P - P^0) \xi^{\perp \beta}|_{C^k(t, y)} \leq \delta_0 |\xi^{\perp \beta}|_{C^{k+m_0}(t, y)} \\ t^{l_2} |(P - P^0) \xi^{\parallel \beta}|_{C^{k,\alpha}(\overline{S}_t)} &\leq \delta_0 |\xi^{\parallel \beta}|_{C^{k+m_0,\alpha}(\overline{S}_t)}, \quad t^{l_2} |(P - P^0) \xi^{\parallel \beta}|_{C^k(t, y)} \leq \delta_0 |\xi^{\parallel \beta}|_{C^{k+m_0}(t, y)} \end{aligned}$$

We say that P satisfies $\mathbb{S}(l)|_{Cyl_{t_0}}$ if it satisfies $\mathbb{S}_\beta(l_1, l_2)|_{Cyl_{t_0}}$ for all β and $l_1 = l_2 = l$. We say that P is $\vec{\beta}$ -*ATID* on N if for any i , it satisfies $\mathbb{S}_{\beta_i}(0, 1)|_{Cyl_{t_0}}$ for some t_0 on the i -th end.

Remark 2.9. By our definition, δ_0 depends on P^0 , B_{P^0} , γ , β , b etc.

Remark 2.10. It's easy to check $\mathbb{S}(l)|_{Cyl_{t_0}}$ for differential operators. In an arbitrary coordinate neighbourhood, write P^0 and P as

$$P^0 = a_{0,1}(y) \frac{\partial}{\partial t} + \Sigma_{\gamma=2}^n a_{0,\gamma}(y) D^\gamma; \quad P = a_1(y, t) \frac{\partial}{\partial t} + \Sigma_{\gamma=2}^n a_\gamma(y, t) D^\gamma. \quad (13)$$

Let δ_1 be small enough with respect to the data in Theorem 3.2 (even smaller than δ_0), then P satisfies $\mathbb{S}(l)|_{Cyl_{t_0}}$ if the following holds for all y , $k \leq k_0 + 1$, $t \geq t_0$.

$$t^l |a_\gamma - a_{0,\gamma}|_{C^k}(t, y) \leq \delta_1 \text{ c.f. [11, (6.5)]}. \quad (14)$$

A simple example of an $\mathbb{S}_0(0, 1)|_{Cyl_{t_0}}$ -operator which does not satisfy (14) is $P^0 + \delta_0 \sin t \frac{\partial}{\partial t}$.

Defining $(\square|_{-\vec{\beta}, \vec{\gamma}, \vec{b}}^{Sobolev, p})$ and $(\square|_{-\vec{\beta}, \vec{\gamma}, \vec{b}}^{Schauder})$ as (2), Theorem 1.4 naturally generalizes to

Theorem 2.11. *Let m_0, k, k_0, α, p be as in Theorem 1.4. Suppose P is $\vec{\beta}$ -ATID elliptic, and β_j is not $P^{0,j}$ -super indicial for any j . Then*

- $(\square|_{-\vec{\beta}, \vec{\gamma}, \vec{b}}^{Sobolev, p})$ is Fredholm if for any j , $b_j \neq 1 - \frac{1}{p}$ or β_j is not $P^{0,j}$ -indicial;
- $(\square|_{-\vec{\beta}, \vec{\gamma}, \vec{b}}^{Schauder})$ is Fredholm if for any j , $b_j \neq 1$ or β_j is not $P^{0,j}$ -indicial.

Dependence of the Constants: we follow the convention in [15, Definition 2.16, 2.17]: the "C" in a result (and the proof) depends on the data in the result, except the " t_0 " (initial time for the cylinders). We will add subscripts when C depends on t_0 or other parameters.

Remark 2.12. From now on, we hide the P^0 (or P) in the $\widehat{W}'s$ ($\widehat{C}'s$). The underlying operator should be clear from the context. When $\gamma = b$, we abbreviate $\widehat{W}_{-\beta, \gamma, b-1}^{k, p, P^0}$, $\widehat{C}_{-\beta, \gamma, b-1}^{k, \alpha, P^0}$, $W_{-\beta, \gamma, b}^{k, p, P^0}$, $C_{-\beta, \gamma, b}^{k, \alpha, P^0}$ to $\widehat{W}_{-\beta, b-1}^{k, p}$, $\widehat{C}_{-\beta, b-1}^{k, \alpha}$, $W_{-\beta, b}^{k, p}$, $C_{-\beta, b}^{k, \alpha}$ respectively. When $\gamma = b = 0$, we further abbreviate $W_{-\beta, b}^{k, p}$, $C_{-\beta, b}^{k, \alpha}$ to $W_{-\beta}^{k, p}$, $C_{-\beta}^{k, \alpha}$.

3 The Local inverses

Definition 3.1. Let $\underline{\beta} < \beta$ be the indicial root adjacent to β from below (but not equal to β), and $\bar{\beta} > \beta$ be the indicial root adjacent to β from above.

Theorem 3.2. *Let P^0 be a TID-operator, and β be not P^0 -super indicial. The following holds in view of Definition 1.1.*

(i): When $b \neq \frac{1}{2}$ or β is not P^0 -indicial, $P^0 : \widehat{W}_{-\beta, \gamma, b-1}^{m_0, 2}(Cyl_{t_0}) \rightarrow L^2_{-\beta, \gamma, b}(Cyl_{t_0})$ admits a bounded linear right inverse. Let $Q_{\beta, +}^{P^0, t_0}$ ($Q_{\beta, -}^{P^0, t_0}$) denote the right inverse when $b > \frac{1}{2}$ ($b < \frac{1}{2}$) respectively when β is indicial, and $Q_{\beta}^{P^0, t_0}$ denote the right inverse when β is not indicial (When β is not indicial, $Q_{\beta, \pm}^{P^0, t_0}$ both mean $Q_{\beta}^{P^0, t_0}$).

(ii): The following (regularity) estimates hold.

$$|Q_{\beta, +}^{P^0, t_0} h|_{\widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}(\overline{Cyl_{t_0}})} \leq C |h|_{C_{-\beta, \gamma, b}^{k, \alpha}(\overline{Cyl_{t_0}})} \text{ when } b > 1 \text{ and } \beta \text{ is indicial}; \quad (15)$$

$$|Q_{\beta, -}^{P^0, t_0} h|_{\widehat{C}_{-\beta, \gamma, 0}^{k+m_0, \alpha}(\overline{Cyl_{t_0}})} \leq C |h|_{C_{-\beta, \gamma, b}^{k, \alpha}(\overline{Cyl_{t_0}})} \text{ when } b > 1 \text{ and } \beta \text{ is indicial}; \quad (16)$$

$$|Q_{\beta, -}^{P^0, t_0} h|_{\widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}(\overline{Cyl_{t_0}})} \leq C |h|_{C_{-\beta, \gamma, b}^{k, \alpha}(\overline{Cyl_{t_0}})} \text{ when } b < 1 \text{ and } \beta \text{ is indicial}; \quad (17)$$

$$|Q_{\beta}^{P^0, t_0} h|_{C_{-\beta, \gamma}^{k+m_0, \alpha}(\overline{Cyl_{t_0}})} \leq C |h|_{C_{-\beta, \gamma}^{k, \alpha}(\overline{Cyl_{t_0}})} \text{ when } \beta \text{ is not indicial}; \quad (18)$$

$$|Q_{\beta, +}^{P^0, t_0} h|_{\widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}(\overline{Cyl_{t_0}})} \leq C |h|_{W_{-\beta, \gamma, b}^{k, p}(\overline{Cyl_{t_0}})} \text{ when } b > 1 - \frac{1}{p} \text{ and } \beta \text{ is indicial}; \quad (19)$$

$$|Q_{\beta, -}^{P^0, t_0} h|_{\widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}(\overline{Cyl_{t_0}})} \leq C |h|_{W_{-\beta, \gamma, b}^{k, p}(\overline{Cyl_{t_0}})} \text{ when } b < 1 - \frac{1}{p} \text{ and } \beta \text{ is indicial}; \quad (20)$$

$$|Q_{\beta}^{P^0, t_0} h|_{W_{-\beta, \gamma}^{k+m_0, p}(\overline{Cyl_{t_0}})} \leq C |h|_{W_{-\beta, \gamma}^{k, p}(\overline{Cyl_{t_0}})} \text{ when } \beta \text{ is not indicial}. \quad (21)$$

(iii): Suppose $\underline{\beta}$ is not super-indicial, then $Q_{\beta,+}^{P^0,t_0} = Q_{\beta,-}^{P^0,t_0}$ on $L^2_{\beta,b}(Cyl_{t_0})$ for any b .

Important Convention: through-out the article, we say that h is in (or not in) a space if and only if the norm of h is $< \infty$ ($= \infty$), respectively. Therefore all the estimates in Theorem 3.2 are regularity estimates.

Theorem 3.3. Suppose P satisfies $\mathbb{S}_{\beta}(0,1)|_{Cyl_{t_0}}$ and β is not P^0 -super-indicial. Then except (16), P also satisfies (i), (ii), (iii) in Theorem 3.2 (with P^0 replaced by P , and $Q_{\beta,+}^{P^0,t_0}$, $Q_{\beta,-}^{P^0,t_0}$, $Q_{\beta}^{P^0,t_0}$ replaced notationally by $Q_{\beta,+}^{P,t_0}$, $Q_{\beta,-}^{P,t_0}$, Q_{β}^{P,t_0}).

Remark 3.4. All the bounds in Theorem 3.2, 3.3 are independent of t_0 .

Proof of Theorem 3.3 assuming 3.2: We momentarily hide t_0, β, \pm in $Q_{\beta,\pm}^{P^0,t_0}$ in each case of Theorem 3.2. Theorem 3.2 and the β -ATID condition (Definition 2.8) implies when δ_0 is sufficiently small, the Neumann-Series (c.f. [17, Theorem 2, page 69])

$$[Id - Q^{P^0}(P^0 - P)]^{-1} \triangleq \sum_{j=0}^{\infty} [Q^{P^0}(P^0 - P)]^j \quad (22)$$

converges to a two-sided inverse of $Id - Q^{P^0}(P^0 - P)$. Hence $Q^P \triangleq (\sum_{j=0}^{\infty} [Q^{P^0}(P^0 - P)]^j) Q^{P^0}$ is a right-inverse of P i.e. $PQ^P = Id$, where we write $P = P^0[Id - Q^{P^0}(P^0 - P)]$. \square

Lemma 3.5. (Hardy's inequality) For any $p \geq 2$,

$$\int_{\frac{1}{10}}^{\infty} (t^{b-1} \int_t^{\infty} |f| ds)^p dt \leq C_{p,b} \int_0^{\infty} (t^b |f|)^p dt \text{ when } b > 1 - \frac{1}{p}; \quad (23)$$

$$\int_{\frac{1}{10}}^{\infty} (t^{b-1} \int_1^t |f| ds)^p dt \leq C_{p,b} \int_0^{\infty} (t^b |f|)^p dt \text{ when } b < 1 - \frac{1}{p}. \quad (24)$$

For all $b \in \mathbb{R}$, $p \geq 2$, $\vartheta \geq 0$, and $\mu \neq 0$, there exists a constant $C_{l_{\mu},b}$ which depends only on b and the lower bound on $|\mu|$ with the following properties.

$$\mu^{p(1+\vartheta)} \int_{\frac{1}{10}}^{\infty} (e^{\mu t} t^b \int_t^{\infty} e^{-\mu s} |f|(s-t)^{\vartheta} ds)^p dt \leq C_{l_{\mu},p,b} \int_0^{\infty} (|f| t^b)^p dt \text{ when } \mu > 0; \quad (25)$$

$$\mu^{p(1+\vartheta)} \int_{\frac{1}{10}}^{\infty} (e^{\mu t} t^b \int_1^t e^{-\mu s} |f|(t-s)^{\vartheta} ds)^p dt \leq C_{l_{\mu},p,b} \int_0^{\infty} (|f| t^b)^p dt \text{ when } \mu < 0. \quad (26)$$

(23) and (24) are special cases of [9, Theorem 330]. The proof for (25), (26) is elementary, we defer it to the Appendix.

The proof of [8, Lemma 6.37, Theorem 7.25] (reflection about the boundary) yields

Claim 3.6. Let $t_0 \geq 2$. For any section $h \in C^{k,\alpha}(Y \times [t_0, t_0 + 1])$ or $W^{k,p}[Y \times (t_0, t_0 + 1)]$, there exists an extension h_{E,t_0} such that

- $h_{E,t_0} = 0$ over $(0, t_0 - 0.01)$, and $h_{E,t_0} = h$ when $t \geq t_0$;
- $|h_{E,t_0}|_{C^{k,\alpha}[Y \times (0, t_0 + 1)]} \leq C|h|_{C^{k,\alpha}[Y \times (t_0, t_0 + 1)]}$, $|h_{E,t_0}|_{W^{k,p}[Y \times (0, t_0 + 1)]} \leq C|h|_{W^{k,p}[Y \times (t_0, t_0 + 1)]}$;
- h_{E,t_0} is translation-invariant in t_0 i.e. $h_{E,t_0}(f)(t) = h_{E,2}(f_{t_0})(t - t_0 + 2)$, where $f_{t_0}(t) = f(t + t_0 - 2)$.

We need to construct a linear operator $\dot{Q}_{\lambda}^{P_{\beta}^0}$ for each of the equations in (11), such that $u_{\lambda} \triangleq \dot{Q}_{\lambda}^{P_{\beta}^0} f_{\lambda}$ solves them respectively with required estimates. Summing the λ 's up, we obtain the desired right inverse:

$$\tilde{Q}_{0,\pm}^{P_{\beta}^0} f \triangleq \sum_{\lambda \in \text{Spec}(B_{P_{\beta}^0})} (\dot{Q}_{\lambda}^{P_{\beta}^0} f_{\lambda}) \phi_{\lambda}. \quad (27)$$

When $\beta \neq 0$, it suffices to take

$$\hat{Q}_{\beta,\pm}^{P^0} \triangleq \sigma_2^{-1} \cdot e^{\beta t} \cdot \tilde{Q}_{0,\pm}^{P_{\beta}^0} \cdot e^{-\beta t} \cdot \sigma_1^{-1}. \quad (28)$$

Proof of Theorem 3.2 (i) for first-order operators: We construct the $\dot{Q}_\lambda^{P^0}$ as

		$u_\lambda (\triangleq \dot{Q}_\lambda^{P^0} f_\lambda)$			$u_\lambda \triangleq \dot{Q}_\lambda^{P^0} f_\lambda$	
1	$\lambda = 0, b > \frac{1}{2}$	$-\int_t^\infty f_\lambda ds$	3	$\lambda > 0, \text{ all } b$	$-e^{\lambda t} \int_t^\infty e^{-\lambda s} f_\lambda ds$	(29)
2	$\lambda = 0, b < \frac{1}{2}$	$\int_1^t f_\lambda ds$	4	$\lambda < 0, \text{ all } b$	$e^{\lambda t} \int_1^t e^{-\lambda s} f_\lambda ds$	

By Remark 2.5 and completeness of the spaces in Definition 2.3, it suffices to assume $f \in C_c^\infty(Cyl_1)$, and only has finitely many non-zero Fourier coefficients. Without loss of generality, we only consider first-order operators, and assume that $\beta = 0$ [see the derivation of (32)]. The proof for second-order operators is similar. (30)

Applying the 4 inequalities in Lemma 3.5 to the 4 cases in (29) respectively, we find

$$\int_1^\infty u_\lambda^2 t^{2b-2} dt \leq C \int_1^\infty f_\lambda^2 t^{2b} dt \text{ (in cases 1, 2), } \lambda^2 \int_1^\infty u_\lambda^2 t^{2\gamma} dt \leq C \int_1^\infty f_\lambda^2 t^{2\gamma} dt \text{ (cases 3, 4).}$$

Using the equation (11) to estimate $\frac{du_\lambda}{dt}$, we trivially obtain

$$\int_1^\infty \left| \frac{du_\lambda}{dt} \right|^2 t^{2b} dt = \int_1^\infty f_\lambda^2 t^{2b} dt \text{ (in cases 1, 2), } \int_1^\infty \left| \frac{du_\lambda}{dt} \right|^2 t^{2\gamma} dt = \int_1^\infty f_\lambda^2 t^{2\gamma} dt \text{ (cases 3, 4).}$$

The above 4 estimates in the 4 cases yield

$$|\tilde{Q}_{0,b}^{P^0} f^{\perp 0}|_{W_{0,\gamma}^{m_0,2}(Cyl_1)} \leq C |f^{\perp 0}|_{L_{0,\gamma}^2(Cyl_1)}; |\tilde{Q}_{0,b}^{P^0} f|_{\widehat{W}_{0,\gamma,b-1}^{m_0,2}(Cyl_1)} \leq C |f|_{L_{0,\gamma,b}^2(Cyl_1)}. \quad (31)$$

Using (28) (σ_i^{-1} are smooth) and the notation in Theorem 3.2 i, let $f = \sigma_1^{-1} h_{E,t_0} \in C_c^\infty(Cyl_1)$, we obtain

$$|\widehat{Q}_{\beta,\pm}^{P^0} h_{E,t_0}|_{\widehat{W}_{-\beta,\gamma,b-1}^{m_0,2}(Cyl_1)} \leq C |h_{E,t_0}|_{L_{-\beta,\gamma,b}^2(Cyl_1)}. \text{ Let } Q_{\beta,\pm}^{P^0,t_0} h \triangleq \widehat{Q}_{\beta,\pm}^{P^0} h_{E,t_0}, \quad (32)$$

the following holds by Claim 3.6.

$$\begin{aligned} |Q_{\beta,\pm}^{P^0,t_0} h|_{\widehat{W}_{-\beta,\gamma,b-1}^{m_0,2}(Cyl_{t_0})} &\leq |\widehat{Q}_{\beta,\pm}^{P^0,t_0} h_{E,t_0}|_{\widehat{W}_{-\beta,\gamma,b-1}^{m_0,2}(Cyl_{t_2})} \leq C |h_{E,t_0}|_{L_{-\beta,\gamma,b}^2(Cyl_1)} \\ &\leq C |h|_{L_{-\beta,\gamma,b}^2(Cyl_{t_0})}. \end{aligned} \quad (33)$$

The above means $Q_{\beta,\pm}^{P^0,t_0}$ is bounded. \square

Similar ideas apply to second-order equations, we defer the detail to the Appendix. By our constructions in (29), (64), (67), (28), we routinely verify Theorem 3.2 (iii).

Proof of Theorem 3.2 (18), (21): It suffices to apply Maz'ya-Plamenevskii's trick ([13, Lemma 1.1, 4.1]). We adopt (30) and assume $k = 0$. Theorem 3.2 (i) yields

$$\begin{aligned} &\int_{l-1}^{l+1} |\tilde{Q}_0^{P^0}(\xi_m f)|_{L^2(Y)}^2 t^{2\gamma} dt \leq C e^{2\mu_0 l} \int_{l-1}^{l+1} |\tilde{Q}_0^{P^0}(\xi_m f)|_{L^2(Y)}^2 e^{-2\mu_0 t} t^{2\gamma} dt \\ &\leq C e^{2\mu_0 l} \int_1^\infty |\xi_m f|_{L^2(Y)}^2 e^{-2\mu_0 t} t^{2\gamma} dt \leq C e^{2\mu_0(l-m)} \int_{m-2}^{m+2} |f|_{L^2(Y)}^2 t^{2\gamma} dt \\ &\leq C e^{-2|\mu_0||l-m|} \sup_m |t^\gamma f|_{L^2(2S_m)}^2, \end{aligned} \quad (34)$$

$$|\tilde{Q}_0^{P^0} f|_{L_{0,\gamma}^2(Cyl_1)} \leq C |f|_{L_{0,\gamma}^2(Cyl_1)}, \quad (35)$$

where we let $|\mu_0|$ be small enough with respect the spectrum gap, and the sign of $-\mu_0$ be the same as that of $l - m$ (when $l = m$ either sign works). Summing the m in (34) over all integers ≥ 3 , we obtain

$$|\tilde{Q}_0^{P_\beta^0} f|_{L_{0,\gamma}^2(S_l)} \leq C \sup_m |t^\gamma f|_{L^2(2S_m)} \sum_{m \geq 2} e^{-|\mu_0||l-m|} \leq C |f|_{C_{0,\gamma}^0(Cyl_1)} \text{ for any } l \geq 2. \quad (36)$$

We recall $|\tilde{Q}_0^{P_\beta^0} f|_{L^2(S_l)} \leq Cl^{-\gamma} |\tilde{Q}_0^{P_\beta^0} f|_{L_{0,\gamma}^2(S_l)}$, $|f|_{C^\alpha(S_l)} \leq Cl^{-\gamma} |f|_{C_{0,\gamma}^\alpha(S_l)}$, $|f|_{L^p(S_l)} \leq Cl^{-\gamma} |f|_{L_{0,\gamma}^p(S_l)}$, and the following standard (Schauder and L^p) estimate on S_l

$$|\xi|_{C^{1,\alpha}(\frac{S_l}{2})} \leq C |P_\beta^0 \xi|_{C^\alpha(S_l)} + C |\xi|_{L^2(S_l)}, \quad |\xi|_{W^{1,p}(\frac{S_l}{2})} \leq C |P_\beta^0 \xi|_{L^p(S_l)} + C |\xi|_{L^2(S_l)}. \quad (37)$$

Then we obtain from (34) and (35) that

$$l^\gamma |\tilde{Q}_0^{P_\beta^0} f|_{C^{1,\alpha}(\frac{S_l}{2})} \leq C [|f|_{C_{0,\gamma}^\alpha(\overline{Cyl_1})} + |f|_{C_{0,\gamma}^0(S_l)}] \leq C |f|_{C_{0,\gamma}^\alpha(\overline{Cyl_1})}, \quad (38)$$

$$l^\gamma |\tilde{Q}_0^{P_\beta^0} f|_{W^{1,p}(\frac{S_l}{2})} \leq C [|f|_{L_{0,\gamma}^p(S_l)} + |\tilde{Q}_0^{P_\beta^0} f|_{L_{0,\gamma}^2(S_l)}]. \quad (39)$$

Taking $\sup_{l \geq 2}$ of (38) and $\sum_{l \geq 2}$ of (39), we obtain by Definition 2.3 and (35) that

$$|\tilde{Q}_0^{P_\beta^0} f|_{C^{1,\alpha}(\overline{Cyl_2})} \leq C |f|_{C_{0,\gamma}^\alpha(\overline{Cyl_1})}, \quad |\tilde{Q}_0^{P_\beta^0} f|_{W_{0,\gamma}^{1,p}(Cyl_2)} \leq C |f|_{L_{0,\gamma}^p(Cyl_1)}. \quad (40)$$

By the same argument in (33) [using (40) instead of (32)], we obtain (18) and (21). \square

Proof of Theorem 3.2 (15), (16), (17), (19), (20): We adopt (30). Using Lemma 3.5 and (29), we find the simple estimates

$$\begin{aligned} |Q_+^{P_\beta^0} f|^0| &\leq C \left| \int_t^\infty f^0 ds \right| \leq C |f|^0|_{C_{0,b}^0(\overline{Cyl_t})} \left| \int_t^\infty s^{-b} ds \right| \leq C |f|^0|_{C_{0,b}^0(\overline{Cyl_t})} t^{1-b} \text{ when } b > 1, \\ |Q_-^{P_\beta^0} f|^0| &\leq C \left| \int_1^t f^0 ds \right| \leq C |f|^0|_{C_{0,b}^0(\overline{Cyl_1})} \left| \int_1^t s^{-b} ds \right| \leq \begin{cases} C |f|^0|_{C_{0,b}^0(\overline{Cyl_1})} t^{1-b} & \text{when } b < 1, \\ C |f|^0|_{C_{0,b}^0(\overline{Cyl_1})} & \text{when } b > 1. \end{cases} \\ \left(\int_1^\infty (t^{b-1} |Q_\pm^{P_\beta^0} f|^0|)^p dt \right)^{\frac{1}{p}} &\leq C \left(\int_1^\infty (t^\gamma |f|^0|)^p dt \right)^{\frac{1}{p}} \text{ when } b > (<) 1 - \frac{1}{p} \text{ respectively.} \end{aligned} \quad (41)$$

Combining $\frac{\partial}{\partial t} Q_\pm^{P_\beta^0} f^0 = f^0$, we find

$$\begin{cases} |Q_+^{P_\beta^0} f|^0|_{\widehat{C}_{0,b-1}^{1,\alpha}(\overline{Cyl_2})} \leq C |f|^0|_{C_{0,b}^\alpha(\overline{Cyl_1})} \text{ when } b > 1 \text{ } (b < 1) \text{ respectively,} \\ |Q_\pm^{P_\beta^0} f|^0|_{\widehat{W}_{0,b-1}^{1,p}(Cyl_2)} \leq C |f|^0|_{L_{0,b}^p(Cyl_1)} \text{ when } b > 1 - \frac{1}{p} \text{ } (b < 1 - \frac{1}{p}) \text{ respectively,} \\ |Q_-^{P_\beta^0} f|^0|_{\widehat{C}_{0,0}^{1,\alpha}(\overline{Cyl_2})} \leq C |f|^0|_{C_{0,b}^\alpha(\overline{Cyl_1})} \text{ when } b > 1. \end{cases} \quad (42)$$

Because $Q_\pm^{P_\beta^0} f^{\perp 0}$ is perpendicular to the kernel, (31) and the proof of (18), (21) yield

$$|\tilde{Q}_{0,\pm}^{P_\beta^0} f^{\perp 0}|_{C_{0,\gamma}^{1,\alpha}(\overline{Cyl_2})} \leq C |f^{\perp 0}|_{C_{0,\gamma}^\alpha(\overline{Cyl_1})}, \quad |\tilde{Q}_{0,\pm}^{P_\beta^0} f^{\perp 0}|_{W_{0,\gamma}^{1,p}(Cyl_2)} \leq C |f^{\perp 0}|_{L_{0,\gamma}^p(Cyl_1)}. \quad (43)$$

(42), (43) amount to (the special cases of) (15), (16), (17), (19), (20) with P^0 replaced by P_β^0 , β by 0. The argument in (32), (33) yields the desired five estimates in general. \square

4 Regularity and proof of Theorem 1.4, 2.11

Remark 4.1. Without loss of generality, in the proof of Claim 4.2, Lemma 4.4, and Proposition 4.5, we only consider first-order operators, and assume $k = t_0 = 1$, $\gamma = b$ (see Remark 2.12). The proof for the other cases is absolutely the same. Though second-order elliptic operators are more complicated [there are 2 homogeneous solutions to the second-order ODEs in (11)], the desired regularity still follows in the same way.

Claim 4.2. *Given any TID-operator P^0 , suppose $\beta, \underline{\beta}$ are not P^0 -super indicial. Then for any $t_0, k \geq 1, \epsilon > 0$, the following estimate holds uniformly in $h \in \ker P^0|_{L^2_{-\beta, \gamma, -\frac{1}{2}+\epsilon}(Cyl_{t_0})}$:*

$$|h|_{C^{k, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{t_0+\epsilon})} \leq C_{t_0} |h|_{L^2_{-\beta, \gamma, -\frac{1}{2}+\epsilon}(Cyl_{t_0})}.$$

Proof: We adopt Remark 4.1. The condition $|h|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)} < \infty$ ($h \in \ker P^0$) implies $h^{\parallel \beta} = 0$, then $h = \sum_{\lambda \in \text{Spec } B_{P^0}, \lambda < \beta} h_\lambda e^{\lambda t} \phi_\lambda = \sum_{\lambda \leq \underline{\beta}} h_\lambda e^{\lambda t} \phi_\lambda$. The condition $\dim \text{Eigen}_{\underline{\beta}} B_{P^0} < \infty$ implies $|h^{\parallel \beta}|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{1+\epsilon})} \leq C |h^{\parallel \beta}|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)}$.

Since $h^{\perp \beta} \in \ker \widehat{B}_{P^0}$, using the Schauder estimate in (37) like (39), the rate of decay is improved i.e. $|h^{\perp \beta}|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{1+\epsilon})} \leq C |h^{\perp \beta}|_{L^2_{-\beta}(Cyl_{1+\frac{\epsilon}{2}})} \leq C |h^{\perp \beta}|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)}$. Thus

$$\begin{aligned} |h|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{1+\epsilon})} &\leq |h^{\perp \beta}|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{1+\epsilon})} + |h^{\parallel \beta}|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_{1+\epsilon})} \\ &\leq C(|h^{\perp \beta}|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)} + |h^{\parallel \beta}|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)}) \\ &\leq C |h|_{L^2_{-\beta, -\frac{1}{2}+\epsilon}(Cyl_1)}. \end{aligned}$$

Definition 4.3. $S_{\beta, \pm}^{P^0, t_0} \triangleq Q_{\beta, \pm}^{P^0, t_0} P^0 - Id$ is bounded from $\widehat{W}_{-\beta, \gamma, -\frac{1}{2} \pm \epsilon}^{m_0, 2}(Cyl_{t_0})$ to itself. We verify $S_{\beta, \pm}^{P, t_0} \triangleq Q_{\beta, \pm}^{P, t_0} P - Id = (\sum_{j=0}^{\infty} [Q_{\beta, \pm}^{P^0, t_0} (P^0 - P)]^j) S_{\beta, \pm}^{P^0, t_0}$.

Lemma 4.4. *Let P be an operator on Cyl_{t_0} as in Definition 2.8. Suppose $\beta, \underline{\beta}$ are not P^0 -super indicial, then the following hold for any $\epsilon > 0, t_0 \geq 2, \gamma$, and $k \leq k_0 + m_0 - 1$.*

$$\left\{ \begin{array}{ll} \text{I:} & |S_{\beta, +}^{P, t_0} \xi|_{\widehat{C}^{k, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_{t_0})} \leq C_{t_0} |\xi|_{\widehat{C}^{k, \alpha}_{-\beta, \gamma, \epsilon}(\overline{Cyl}_{t_0})} \text{ when } P \text{ satisfies } \textcircled{S}(l)|_{Cyl_{t_0}} \text{ with } l > 1, \\ \text{II:} & |S_{\beta, +}^{P, t_0} \xi|_{\widehat{W}_{-\underline{\beta}, -\frac{1}{2}-\epsilon}^{m_0, 2}(Cyl_{t_0})} \leq C_{t_0} |\xi|_{\widehat{W}_{-\beta, \gamma, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})} \text{ when } P \text{ satisfies } \textcircled{S}_{\beta}|_{Cyl_{t_0}}, \\ \text{III:} & |S_{\beta, +}^{P, t_0} \xi|_{\widehat{C}^{k, \alpha}_{-\underline{\beta}, -\epsilon}(\overline{Cyl}_{t_0})} \leq C_{t_0} |\xi|_{\widehat{C}^{k, \alpha}_{-\beta, \gamma, \epsilon}(\overline{Cyl}_{t_0})} \text{ when } P \text{ satisfies } \textcircled{S}_{\beta}|_{Cyl_{t_0}}, \\ \text{IV:} & |S_{\beta, +}^{P, t_0} \xi|_{C^{k, \alpha}_{-\underline{\beta}, -\frac{1}{2}-\epsilon}(\overline{Cyl}_{t_0+\epsilon})} \leq C_{t_0} |\xi|_{\widehat{W}_{-\beta, \gamma, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})} \text{ when } P \text{ satisfies } \textcircled{S}_{\beta}|_{Cyl_{t_0}}. \end{array} \right.$$

Proof. We adopt Remark 4.1 and only prove **I** and **IV**. **II** and **III** are similar (to **I**). First, we deal with the model operator $S_{\beta, +}^{P^0, t_0}$. Because $S_{\beta, +}^{P^0, 2} \xi \in \ker P^0$, Claim 4.2 yields

$$\begin{aligned} |S_{\beta, +}^{P^0, 2} \xi|_{\widehat{C}^{1, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_2)} &\leq C |S_{\beta, +}^{P^0, 2} \xi|_{\widehat{C}^{1, \alpha}_{-\beta, \epsilon}(Y \times [2, 4])} + |S_{\beta, +}^{P^0, 2} \xi|_{\widehat{C}^{1, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_3)} \\ &\leq C |S_{\beta, +}^{P^0, 2} \xi|_{\widehat{C}^{1, \alpha}_{-\beta, \epsilon}(\overline{Cyl}_2)} \leq C |Q_{\beta, +}^{P^0, 2} P^0 \xi|_{\widehat{C}^{1, \alpha}_{-\beta, \epsilon}(\overline{Cyl}_2)} + C |\xi|_{\widehat{C}^{1, \alpha}_{-\beta, \epsilon}(\overline{Cyl}_2)} \\ &\leq C |\xi|_{\widehat{C}^{1, \alpha}_{-\beta, \epsilon}(\overline{Cyl}_2)}. \end{aligned} \tag{44}$$

When P satisfies $\textcircled{S}(l)|_{Cyl_1}$ with $l > 1$, $|(P^0 - P)\eta|_{C^{\alpha}_{-\underline{\beta}, l}(\overline{Cyl}_2)} \leq C \delta_0 |\eta|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_2)}$ for any η . Hence we obtain the following (by Theorem 3.2 (16) and the above).

$$|Q_{\beta, +}^{P^0, 2} (P^0 - P)\eta|_{\widehat{C}^{1, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_2)} = |Q_{\beta, -}^{P^0, 2} (P^0 - P)\eta|_{\widehat{C}^{1, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_2)} \leq C \delta_0 |\eta|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_2)}. \tag{45}$$

Let δ_0 be small enough such that $C \delta_0 \leq \frac{1}{2}$ in the above, the Neumann series converges i.e.

$$|(\sum_{j=0}^{\infty} [Q_{\beta, +}^{P^0, 2} (P^0 - P)]^j) \eta|_{\widehat{C}^{1, \alpha}_{-\underline{\beta}, 0}(\overline{Cyl}_2)} \leq C |\eta|_{C^{1, \alpha}_{-\underline{\beta}}(\overline{Cyl}_2)}. \tag{46}$$

Let $\eta = S_{\beta, +}^{P^0, 2} \xi$ in (46), Lemma 4.4 **I** follows from (44) and Definition 4.3.

On **IV**, using the Schauder estimate (37) and **II** (note $S_{\beta, +}^{P, 2} \xi \in \ker P$), we obtain

$$|S_{\beta, +}^{P, 2} \xi|_{C^{k, \alpha}(\epsilon S_L)} \leq C |S_{\beta, +}^{P, 2} \xi|_{L^2(\epsilon S_L)} \leq C l^{\frac{1}{2}+\epsilon} e^{\beta l} |S_{\beta, +}^{P, 2} \xi|_{L^2_{-\underline{\beta}, -\frac{1}{2}-\epsilon}(\epsilon S_L)}. \tag{47}$$

By Definition 2.7, the proof of **IV** is complete. \square

Proposition 4.5. *Under the same conditions on P , ϵ , k , γ , β , $\underline{\beta}$ in Lemma 4.4, the following hold uniformly for any $h \in \text{Ker} P|_{\widehat{W}_{-\beta, \gamma, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})}$.*

$$\begin{cases} |h|_{\widehat{C}_{-\underline{\beta}, -\epsilon}^{k, \alpha}(\overline{Cyl}_{t_0+\epsilon})} \leq C_{t_0} |h|_{\widehat{W}_{-\beta, \gamma, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})} & \text{when } P \text{ satisfies } \mathbb{S}_{\beta}|_{Cyl_{t_0}}, \\ |h|_{\widehat{C}_{-\underline{\beta}, 0}^{k, \alpha}(\overline{Cyl}_{t_0+\epsilon})} \leq C_{t_0} |h|_{\widehat{W}_{-\beta, \gamma, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})} & \text{when } P \text{ satisfies } \mathbb{S}(l)|_{Cyl_{t_0}} \text{ with } l > 1. \end{cases}$$

Consequently, for any $k \leq k_0 - 2$, suppose the signs of μ_1 and μ_2 are the same, we have

$$(\text{ker}|\text{Coker}| \text{Index})(\square|_{-\beta, \gamma, 1-\frac{1}{p}+\mu_1}^{\text{Sobolev}, p}) = (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\beta, \gamma, 1+\mu_2}^{\text{Schauder}}) \text{ respectively.} \quad (48)$$

Moreover, suppose $\underline{\beta} < \overline{\beta}$ are 2 adjacent indicial roots which are not P^0 -super indicial, then

$$\begin{aligned} & (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_1, 1-\frac{1}{p}+\epsilon_1}^{\text{Sobolev}, p}) = (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_2, b}^{\text{Sobolev}, p}) \\ & = (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_3, 1-\frac{1}{p}-\epsilon_2}^{\text{Sobolev}, p}). \\ & (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_1, 1+\epsilon_1}^{\text{Schauder}}) = (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_2, b}^{\text{Schauder}}) \\ & = (\text{ker}|\text{Coker}| \text{Index})(\square|_{-\underline{\beta}, \gamma_3, 1-\epsilon_2}^{\text{Schauder}}). \end{aligned}$$

for any $p \geq 2$, b , γ_i ($i = 1, 2, 3$), $\beta \in (\underline{\beta}, \overline{\beta})$, and $\epsilon_1, \epsilon_2 > 0$.

Proof: It's a direct corollary of Lemma 4.4. We only prove the second assertion, the first is easier. We adopt Remark 4.1. We note that $h \in \text{Ker} P$ implies

$$h = -S_{\beta, +}^{P, t} h \text{ for all } t \geq t_0. \quad (49)$$

Let $t = t_0$, Lemma 4.4 (IV) says $|h|_{\widehat{C}_{-\underline{\beta}, -2}^{k, \alpha}(\overline{Cyl}_{t_0+\epsilon})} \leq C|h|_{\widehat{W}_{-\beta, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})}$. Let $t = t_0 + \epsilon$ in (49), Lemma 4.4 (I) and the above imply

$$|h|_{\widehat{C}_{-\underline{\beta}, 0}^{k, \alpha}(\overline{Cyl}_{t_0+\epsilon})} \leq C|h|_{\widehat{C}_{-\underline{\beta}, -2}^{k, \alpha}(\overline{Cyl}_{t_0+\epsilon})} \leq C|h|_{\widehat{W}_{-\beta, -\frac{1}{2}+\epsilon}^{m_0, 2}(Cyl_{t_0})}.$$

Proof of Theorem 1.4, 2.11: With the help of Lemma 4.4, it is standard. We only do the argument for 1.4, and only show the \widehat{W} (\widehat{C}) theory for $b > 1 - \frac{1}{p}$ ($b > 1$), respectively.

By [11, (2.5) to line 2, page 421] (also see [15, Theorem 4.14]), piecing together the local right inverses in Theorem 3.3 on the ends, we obtain a global parametrix $Q_{\beta, +, N}$ such that

$$\begin{cases} Q_{\beta, +, N} P = Id + S_{left}, \\ P Q_{\beta, +, N} = Id + S_{right} \end{cases}, \quad S_{right} \text{ is compact from } W_{-\beta, \gamma, b}^{k, p}(C_{-\beta, \gamma, b}^{k, \alpha}) \text{ to itself.} \quad (50)$$

Lemma 4.4 IV says S_{left} is bounded from $\widehat{W}_{-\beta, \gamma, b-1}^{m_0, 2}$ to $\widehat{C}_{-\underline{\beta}, -1}^{k_0+m_0-1, \alpha}$. Since the weight is improved, by the (cylindrical analogue of) [15, proof of Lemma 4.10], $\widehat{C}_{-\underline{\beta}, -1}^{k_0+m_0-1, \alpha}$ embeds compactly into $\widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}$ and $\widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}$ when $k \leq k_0 - 2$. Then S_{left} is compact from $\widehat{W}_{-\beta, \gamma, b-1}^{k+m_0, p}$ ($\widehat{C}_{-\beta, \gamma, b-1}^{k+m_0, \alpha}$) to itself. Hence [18, Theorem 4.6.5] implies P is Fredholm. \square

5 Index

Definition 5.1. Let B be an operator as in Proposition 5.4. Let $d_\lambda \triangleq \dim \text{ker}(B - \lambda Id)$. Let $\text{Eta}(B)$ denote the eta-invariant defined in [3, Theorem 3.10 (iii)], and

$$H_{\beta, B} \triangleq \begin{cases} -\frac{d_0}{2} - \sum_{0 < \lambda \leq \beta} d_\lambda & \text{when } \beta < 0, \\ -\frac{d_0}{2} & \text{when } \beta = 0, \\ \frac{d_0}{2} + \sum_{0 < \lambda < \beta} d_\lambda & \text{when } \beta > 0. \end{cases}, \quad H_{\beta, B, b} \triangleq \begin{cases} H_{\beta, B} & \text{when } b > \frac{1}{2}, \\ H_{\beta, B} + d_\beta & \text{when } b < \frac{1}{2}. \end{cases}$$

Theorem 5.2. *Under the conditions in Theorem 2.11, suppose in addition that P is first-order elliptic, σ_1 in (1) is an isometry, and P is translation invariant on each end. Then*

$$\text{index } P|_{\widehat{W}_{-\vec{\beta}, \vec{b}-1}^{1,2} \rightarrow L_{-\vec{\beta}, \vec{b}}^2} = \int_X \alpha_0 d\text{vol} - \frac{\text{Eta}(B_{P^0})}{2} + \sum_{j=1}^{l_0} H_{\beta_j, B_{P^0, j}^j, b_j}, \text{ where} \quad (51)$$

α_0 is the 0-th order term in the expansion of the kernel of $e^{-t\tilde{P}^* \tilde{P}} - e^{-t\tilde{P} \tilde{P}^*}$, \tilde{P} is the double of P on the double of N_0 ($N_0 \# \bar{N}_0$).

Remark 5.3. When P satisfies proper conditions as Definition 2.8, under the operator norm, we can usually deform it continuously to be translation-invariant on each end. Thus the index of P can still be computed by deformation invariance.

Proof of Theorem 5.2: Without loss of generality, we assume $\sigma_2 = Id$. We recall the "Extended L^2 -sections" defined in [3, first paragraph of page 58], and note that the dual of $L_{0,b}^2$ is isomorphic to $L_{0,-b}^2$. Using Claim 4.2 for both L and $L^* = -(\frac{\partial}{\partial t} + B)\sigma_1^{-1}$, we have

$$\begin{aligned} \text{Ker } L|_{\widehat{W}_{0,b-1}^{1,2}} &= \text{Ker } L|_{L^2(N,E)}, \quad \text{Ker } L^*|_{L_{0,-b}^2} = \text{Ker } L^*|_{\text{Extended } L^2(N,F)} \text{ when } b > \frac{1}{2}; \\ \text{Ker } L|_{\widehat{W}_{0,b-1}^{1,2}} &= \text{Ker } L|_{\text{Extended } L^2(N,E)}, \quad \text{Ker } L^*|_{L_{0,-b}^2} = \text{Ker } L^*|_{L^2(N,F)} \text{ when } b < \frac{1}{2}. \end{aligned} \quad (52)$$

$$\text{Index } L|_{\widehat{W}_{0,b-1}^{1,2} \rightarrow L_{0,b}^2} = \begin{cases} h(E) - h(F) - h_\infty(F) & \text{when } b > \frac{1}{2}; \\ h(E) - h(F) + h_\infty(E) & \text{when } b < \frac{1}{2}, \end{cases} \quad (53)$$

where $h(E)$, $h(F)$, $h_\infty(F)$, $h_\infty(E)$ are defined in [3, Corollary (3.14)]. Assuming $\vec{\beta} = 0$ for all j , (51) follows from [3, Corollary (3.14), (3.25)].

By Proposition 4.5 (c.f. Remark 1.7),

$$\text{Index}(P|_{-\beta, b}^{\text{Sobolev}, 2}) = \begin{cases} \text{Index}(P|_{-\beta+\epsilon}^{\text{Sobolev}, 2}) & \text{when } b > \frac{1}{2}; \\ \text{Index}(P|_{-\beta-\epsilon}^{\text{Sobolev}, 2}) & \text{when } b < \frac{1}{2}. \end{cases} \quad \epsilon > 0 \text{ and small.} \quad (54)$$

The index change formula of Lockhart-McOwen [11, Theorem 8.1] means for any i_0 , let β_i , $i \neq i_0$ be unaltered, and β_{i_0} go across an eigen-value λ of B_{i_0} (from $\lambda + \epsilon$ to $\lambda - \epsilon$), the index decreases by $\dim \ker(B_{i_0}^{i_0} - \lambda Id)$. The proof for general $\vec{\beta}$ is complete with the help of (54) [and (51) for $\vec{\beta} = 0$]. \square

Proposition 5.4. *In the setting of Definition 1.1, suppose B is a self-adjoint first-order elliptic differential operator on $E \rightarrow Y$. Then $\text{Eta}(B - \beta Id) = \text{Eta}(B) - 2H_{\beta, B} - d_\beta$.*

Proof. We form the full cylinder $Y \times (-\infty, +\infty)_t$ and consider $P = \frac{\partial}{\partial t} - B$. We consider the open cover End_+ , $Y \times (-4, 4)$, End_- , where $\text{End}_+ = \text{Cyl}_3$ and $\text{End}_- = Y \times (-\infty, -3)$. In End_- , under the coordinate $s = -t$, $P = -\frac{\partial}{\partial s} - B = -[\frac{\partial}{\partial s} - (-B)]$. Let $-\vec{\beta} = (-\beta, 0)$ ($e^{-\beta t}$ on End_+ and 1 on End_-), Theorem 5.2 says when $b > \frac{1}{2}$ that

$$\text{index } P|_{\widehat{W}_{-\vec{\beta}, b-1}^{1,2} \rightarrow L_{-\vec{\beta}, b}^2} = \int_X \alpha_0 d\text{vol} - \frac{\text{Eta}(B)}{2} - \frac{\text{Eta}(-B)}{2} - \frac{d_0}{2} + H_{\beta, B}. \quad (55)$$

On the other hand, let ρ be a smooth function which is equal to $e^{\beta t}$ on End_+ , and 1 on End_- , the conjugation $P_\rho = \frac{1}{\rho} \cdot P \cdot \rho : \widehat{W}_{0, b-1}^{1,2} \rightarrow L_{0, b}^2$ has the same index as P . Noting

$$P_\rho = \begin{cases} \frac{\partial}{\partial t} - (B - \beta Id) & \text{on } \text{End}_+ \\ P & \text{on } \text{End}_-, \end{cases}$$

then Theorem 5.2 says the following for P_ρ .

$$\text{index } P_\rho|_{\widehat{W}_{0, b-1}^{1,2} \rightarrow L_{0, b}^2} = \int_X \alpha_0 d\text{vol} - \frac{\text{Eta}(B - \beta Id)}{2} - \frac{\text{Eta}(-B)}{2} - \frac{d_\beta}{2} - \frac{d_0}{2}. \quad (56)$$

The equality between (55) and (56) yields the desired identity. \square

6 Applications

Definition 6.1. Let Γ be a section or connection of $E \rightarrow \text{Cyl}_0$. Suppose Γ satisfies an elliptic equation of order m_0 . For any $C_0 > 0$, $\tau \geq 0$, and real number b , we say that Γ converges to Γ_0 (at least) at the rate $\frac{C_0 e^{-\tau t}}{t^b}$ [or $O(\frac{e^{-\tau t}}{t^b})$], if Γ_0 is a section or connection of $E \rightarrow Y$ respectively and $\limsup_{t \rightarrow \infty} \frac{e^{-\tau t}}{t^b} |\Gamma - \Gamma_0|_{C^{m_0, \alpha}(S_t)} < C_0$ (or $< \infty$) respectively.

When $\tau > 0$, We say that the convergence is exponential. When $\tau = 0$, We say that it's polynomial. We only prove 1.11 in full detail, 1.9 follows similarly.

6.1 Yang-Mills connections: proof for Corollary 1.11

Denoting $A - A_O$ by a , the YM-equation is $0 = d_A^* F_A = d_{A_O+a}^* (d_{A_O} a + F_{A_O} + \frac{1}{2}[a, a])$.

Thus $\Delta_{A_O, \text{Hodge}} a + (-1)^{n+1} \star [a, \star F_{A_O}] = Q_{YM}(a) - d_{A_O}^* F_{A_O}$ assuming $d_{A_O}^* a = 0$, (57)

where $\Delta_{A_O, \text{Hodge}} = d_{A_O}^* d_{A_O} + d_{A_O} d_{A_O}^*$ is the Hodge Laplacian, and

$$Q_{YM}(a) = -\frac{d_{A_O}^* [a, a]}{2} + (-1)^n \star [a, \star d_{A_O} a] + \frac{(-1)^n}{2} \star [a, \star [a, a]]. \quad (58)$$

We note Q_{YM} is quadratic in a . We routinely verify

$$(-1)^{n+1} \star [a, \star F_{A_O}] = F_{A_O} \otimes a \quad ([15, \text{Definition 3.2}], \Delta_{A_O, \text{Hodge}, g_E} = \nabla_{A_O}^{\star E} \nabla_{A_O} + F_{A_O} \otimes_{g_E} a).$$

Then in cylindrical coordinates ($r = e^{-t}$), let $a = v dt + \theta$ (θ does not contain dt), we find

$$\begin{aligned} \Delta_{A_O, \text{Hodge}, E} a + (-1)^{n+1} \star_E [a, \star_E F_{A_O}] &= \nabla_{A_O}^{\star E} \nabla_{A_O} + 2F_{A_O} \otimes_{g_E} a \\ &= -e^{2t} \begin{vmatrix} dt & 0 \\ 0 & Id \end{vmatrix} \left\{ \frac{\partial^2}{\partial t^2} - (n-4) \frac{\partial}{\partial t} - B \right\} \begin{bmatrix} v \\ \theta \end{bmatrix}. \quad \text{Let } Y = \mathbb{S}^{n-1}, \quad B \text{ is} \quad (59) \\ B \begin{bmatrix} v \\ \theta \end{bmatrix} &\triangleq \begin{bmatrix} \nabla^{\star_Y} \nabla v - 2d^{\star} b - 2(n-2)v \\ \nabla^{\star_Y} \nabla \theta - 2dv - 2F_{A_O} \otimes_Y \theta - (n-2)\theta \end{bmatrix} \quad (\text{c.f. } [15, (22), (24)]). \quad (60) \end{aligned}$$

As an usual strategy for non-linear equations, we view Q_{YM} as a linear operator defining

$$\widehat{Q}_{YM, a}(b) = -\frac{d_{A_O}^* [b, a]}{2} + (-1)^n \star [b, \star d_{A_O} a] + \frac{(-1)^n}{2} \star [b, \star [a, a]]. \quad (61)$$

Hence $Q_{YM}(a) = \widehat{Q}_{YM, a}(a)$, and we can write (57) in cylindrical coordinates as

$$P_{YM}(a) \triangleq \{e^{-2t}(\Delta_{A_O, \text{Hodge}} + (-1)^{n+1} \star [\cdot, \star F_{A_O}]) - e^{-2t} \widehat{Q}_{YM, a}\} a = -e^{-2t} d_{A_O}^* F_{A_O}. \quad (62)$$

The conditions on g , a , (59), and (60) implies P_{YM} is ATID in the cylindrical coordinates. Moreover, that $d_{A_O}^{\star E} F_{A_O} = 0$ (tangent connection is Yang-Mills) implies $-e^{-2t} d_{A_O}^* F_{A_O} \in C_{1,0}^\alpha(\overline{\text{Cyl}}_{-\log R})$ (exponential decay). Applying $Q_{0,+}^{P_{YM}, -2 \log R} (= Q_{0,-}^{P_{YM}, -2 \log R})$ to both sides of (62), Lemma 4.4 **III** implies that a decays exponentially. This in turn means P_{YM} satisfies $\textcircled{S}(l)$ for all $l > 1$. Then Corollary 1.11 *I* follows from applying Proposition 4.5 to $a + Q_{0,+}^{P_{YM}, -2 \log R} e^{-2t} d_{A_O}^* F_{A_O} \in \text{Ker } P_{YM}$ and Theorem 3.3 to $Q_{0,+}^{P_{YM}, -2 \log R} e^{-2t} d_{A_O}^* F_{A_O}$.

Combining *I* and Lemma 7.2, the proof of *II* is complete.

6.2 Minimal Surfaces: proof for Corollary 1.9

By [1, Section 5, page 247], in the cylindrical coordinate $t = \log \frac{1}{|x|}$, the graph type minimal sub-manifold equation can be written as (63) in terms of a section u to $T^\perp \Sigma|_{\mathbb{S}^N}$ (the normal bundle of Σ in \mathbb{S}^N). We note (the transition functions of) $T^\perp \Sigma|_{\mathbb{S}^N}$ does not depend on t (or $|x|$), then this bundle is in the case considered by Definition 1.1.

$$P_{MSM} u \triangleq Lu + \mathfrak{R}(u) + (M_\Sigma - L_\Sigma)u = 0, \quad \text{where} \quad (63)$$

- $Lu = u'' - (n+1)u' + L_\Sigma u$ (see [16, Page 565] and [1, (1.8)]),
- $\mathfrak{R}(u)$ satisfies [16, (1.9)], L_Σ is the linearisation of M_Σ (see [1, (5.2) and Page 248]).

By the idea in (61), we can view $\mathfrak{R} + M_\Sigma - L_\Sigma$ as a linear operator i.e.

$$(\mathfrak{R} + M_\Sigma - L_\Sigma)(v) \triangleq Q_{MSM}(v) = a(x, t, u, \nabla u, \nabla^2 u, u') \cdot \nabla^2 v + b(x, t, u, \nabla u, \nabla^2 u) \cdot v' \\ + c(x, t, u, \nabla u, \nabla^2 u) \cdot \nabla v' + d(x, t, u, \nabla u, \nabla^2 u) \cdot v''$$

where a, b, c, ∇ can be found in [16, the paragraph enclosing (1.9); between line 1 in page 565 and (7.35)]. Exactly as the proof of Corollary 1.11, u decays as $\frac{\delta_0}{t}$ implies P_{MSM} is 0-ATID. Thus Proposition 4.5 [and the discussion below (62)] yields the desired improvement.

7 Appendix

Proof of Theorem 3.2 (i) for second-order operators: We mainly focus on the case when $a_1 - 2\beta > 0$. We solve the second-order ODE in (11) according to the following.

	$m \triangleq a_1 - 2\beta$. Assume $m > 0$.	Corresponding solution $u_\lambda \triangleq \dot{Q}_\lambda^{P_0} f_\lambda$ and the derivative. $Det \triangleq m^2 + 4\lambda$, $\mu \triangleq \frac{\sqrt{ Det }}{2}$, $\mu^+ \triangleq \frac{m}{2} + \mu$, $\mu^- \triangleq \frac{m}{2} - \mu$	
1	$Det > 0, \lambda \neq 0$	$u_\lambda = -\frac{1}{\sqrt{Det}}[e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds + e^{\mu^- t} \int_1^t e^{-\mu^- s} f_\lambda ds]$, $u'_\lambda = -\frac{1}{\sqrt{Det}}[\mu^+ e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds + \mu^- e^{\mu^- t} \int_1^t e^{-\mu^- s} f_\lambda ds]$	
2	$\lambda = 0, b > \frac{1}{2}$	$u_\lambda = \frac{1}{\sqrt{Det}}\{-e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds + \int_t^\infty f_\lambda ds\}$, $u'_\lambda = -\frac{\mu^+}{\sqrt{Det}}e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds$	(64)
3	$\lambda = 0, b < \frac{1}{2}$	$u_\lambda = \frac{1}{\sqrt{Det}}\{-e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds - \int_1^t f_\lambda ds\}$, $u'_\lambda = -\frac{\mu^+}{\sqrt{Det}}e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds$	
4	$Det = 0$	$u_\lambda = e^{\frac{mt}{2}} \int_t^\infty e^{-\frac{ms}{2}} f_\lambda(s-t) ds$	
5	$Det < 0$	$u_\lambda = \frac{e^{\frac{mt}{2}}}{\mu} \{\int_t^\infty e^{-\frac{ms}{2}} f_\lambda [\cos \mu t \sin(\mu s) - \sin \mu t \cos(\mu s)] ds\}$	

Case 1, 5 in (64): When $\lambda \neq 0$, we have $\frac{\sqrt{|\lambda|}}{C} \leq \mu^+ \leq C\sqrt{|\lambda|}$, $-\frac{\sqrt{|\lambda|}}{C} \leq \mu^- \leq -C\sqrt{|\lambda|}$. We use (25), (26) to estimate u_λ and u'_λ termwise, then use the second-order equation in (11) to estimate u''_λ . Then we obtain the following for Case 1 and 4.

$$\int_1^\infty |u''_\lambda|^2 t^{2\gamma} dt + |\lambda| \int_1^\infty |u'_\lambda|^2 t^{2\gamma} dt + \lambda^2 \int_1^\infty u_\lambda^2 t^{2\gamma} dt \leq C \int_1^\infty f_\lambda^2 t^{2\gamma} dt. \quad (65)$$

Since $|\cos x|, |\sin x| \leq 1$ for all x , and $\frac{1}{\sqrt{\lambda}} \leq C$ when $\lambda \neq 0$ [$Spec(B)$ is discrete], (65) holds for Case 5 in similar way.

Cases 2,3 in (64): We still do the termwise estimates. Using (25) for $e^{\mu^+ t} \int_t^\infty e^{-\mu^+ s} f_\lambda ds$, (26) for $e^{-\mu^+ t} \int_1^t e^{\mu^+ s} f_\lambda ds$, and (23) for $\int_t^\infty f_\lambda ds$, we obtain

$$\int_1^\infty |u''_\lambda|^2 t^{2b} dt + \int_1^\infty |u'_\lambda|^2 t^{2b} dt + \int_1^\infty u_\lambda^2 t^{2b-2} dt \leq C \int_1^\infty f_\lambda^2 t^{2b} dt. \quad (66)$$

Remark 2.5, (65), (66) amount to $|\tilde{Q}_{0,b}^{P_0} f|_{\widehat{W}_{0,\gamma,b-1}^{2,2}(Cy_{l_1})} \leq C \|f\|_{L_{0,\gamma,b}^2(Cy_{l_1})}$. The argument between (31) and (33) gives the desired local right inverse $Q_{\beta,b}^{P_0,t_0}$ with the desired bound.

The proof when $a_1 - 2\beta < 0$ is by the same analysis, but to the following table of solutions [using the definitions in the first row of (64) but assuming $m = a_1 - 2\beta < 0$].

1	$Det > 0, \lambda \neq 0$	the same as Case 1 in (64)	
2	$\lambda = 0, b > \frac{1}{2}$	$u_\lambda = \frac{1}{\sqrt{Det}}\{-\int_t^\infty f_\lambda ds - e^{\mu^- t} \int_1^t e^{-\mu^- s} f_\lambda ds\}$	
3	$\lambda = 0, b < \frac{1}{2}$	$u_\lambda = \frac{1}{\sqrt{Det}}\{-e^{\mu^- t} \int_1^t e^{-\mu^- s} f_\lambda ds + \int_1^t f_\lambda ds\}$	(67)
4	$Det = 0$	$u_\lambda = e^{\frac{mt}{2}} \int_1^t e^{-\frac{ms}{2}} f_\lambda(t-s) ds$	
5	$Det < 0$	$u_\lambda = -\frac{e^{\frac{mt}{2}}}{\mu} \{\int_1^t e^{-\frac{ms}{2}} f_\lambda [\cos \mu t \sin(\mu s) - \sin \mu t \cos(\mu s)] ds\}$	

The proof when $m = a_1 - 2\beta = 0$ and $\lambda > 0$ is much easier: we only need to use (exactly) the solutions in Case 1 of (67). Case 2–5 are super-indicial (which we don't consider). \square

In the cone setting as Corollary 1.11, let $\nabla_{A_O}^* \nabla_{A_O}$ denote the rough Laplacian on $\Omega^0(adE)$. By [15, (17)], in cylindrical coordinate $t = -\log r$,

$$\nabla_{A_O}^* \nabla_{A_O} \chi = e^{2t} \left\{ \frac{d^2 \chi}{dt^2} - (n-2) \frac{d\chi}{dt} + \Delta_{A_O, \mathbb{S}^{n-1}} \chi \right\}, \quad (68)$$

where $\Delta_{A_O, \mathbb{S}^{n-1}}$ is the rough Laplacian on the link. Let $C_{\gamma, b, cone}^{k, \alpha}$ denote the weighted Schauder-spaces in the cone setting defined in [15, Definition 2.10]. On 0-forms, we note that pulling back is an isomorphism $C_{\gamma, b, cone}^{k, \alpha}[B_O(R)] \rightarrow C_{-\gamma, b}^{k, \alpha}(Cyl_{-\log R})$.

Definition 7.1. We say that A_O is irreducible if every parallel section to $\Omega^0(adE)$ is 0 everywhere. This means 0 is not an eigenvalue of $\Delta_{A_O, \mathbb{S}^{n-1}}$.

Lemma 7.2. (*Existence of Coulomb gauge*) Under the setting in the first paragraph of Corollary 1.11 (for any $n \geq 3$), suppose A_O is irreducible. Suppose A is a C^2 -connection on $B_O(R) \setminus O$ such that

$$|A - A_O|_{C_{1, b, cone}^{1, \alpha}[B_O(R)]} < \delta_0 \text{ for some } b \geq 0. \quad (69)$$

Then there exists a gauge s on $B_O(R)$ such that

- $|s - Id|_{C_{0, b, cone}^{2, \alpha}[B_O(R)]} \leq C|A - A_O|_{C_{1, b, cone}^{1, \alpha}[B_O(R)]} \leq C\delta_0$,
- $d_{A_O}^*[s(A) - A_O] = 0$ in a smaller (truncated) ball $B_O(R') \setminus O$, and

$$|S(A) - A_O|_{C_{1, b, cone}^{1, \alpha}[B_O(R') - O]} \leq C|A - A_O|_{C_{1, b, cone}^{1, \alpha}[B_O(R) - O]} \leq C\delta_0. \quad (70)$$

Proof. Let $a \triangleq A - A_O$. By Theorem 3.3 and paragraph enclosing (68), for any $b \geq 0$ and R' small enough, $\nabla_A^* \nabla_A$ is invertible: $C_{0, b, cone}^{2, \alpha}[B(R')]|_{\Omega^0(adE)} \rightarrow C_{2, b, cone}^\alpha[B(R')]|_{\Omega^0(adE)}$. Writing $s = e^{-\chi}$, then

$$d_{A_O}^*[s(A) - A_O] = d_{A_O}^*[s^{-1}as + s^{-1}d_{A_O}s] = d_{A_O}^*[e^{-\chi}ae^\chi + e^{-\chi}d_{A_O}e^\chi] \quad (71)$$

is a continuously differentiable map:

$$C_{1, b, cone}^{1, \alpha}[B_O(R)]|_{\Omega^1(adE)} \times C_{0, b, cone}^{2, \alpha}[B_O(R)]|_{\Omega^0(adE)} \rightarrow C_{2, b, cone}^\alpha[B_O(R)]|_{\Omega^0(adE)}.$$

Then the proof is complete by the standard argument in [6, Proposition 2.3.4]. \square

Proof of (25), (26): For any $s \geq \frac{1}{10}$, leaving the proof to the readers, we have

$$|\int_1^s e^{\mu t} t^d dt| \leq C_{l_\mu, d} \frac{e^{\mu s} s^d}{\mu} \text{ when } \mu > 0, \quad \int_s^\infty e^{\mu t} t^d dt \leq C_{l_\mu, d} \frac{e^{\mu s} s^d}{-\mu} \text{ when } \mu < 0. \quad (72)$$

For (25), Hölder's inequality and the change of variable $z = s - t$ yield

$$\begin{aligned} & \left(\int_t^\infty e^{-\mu s} f(s-t)^\vartheta ds \right)^p \leq \left(\int_t^\infty e^{-\mu s} f^p ds \right) \left(\int_t^\infty e^{-\mu s} (s-t)^{\frac{p\vartheta}{p-1}} ds \right)^{p-1} \\ & \leq C \left(\int_t^\infty e^{-\mu s} f^p ds \right) \frac{e^{-\mu(p-1)t}}{\mu^{\vartheta p + p - 1}}. \end{aligned} \quad (73)$$

Then (72) and (73) yield

$$\begin{aligned} & \int_{\frac{1}{10}}^\infty (e^{\mu t} t^b \int_t^\infty e^{-\mu s} f(s-t)^\vartheta ds)^p dt \leq \frac{1}{\mu^{p\vartheta + p - 1}} \left(\int_{\frac{1}{10}}^\infty e^{-\mu s} f^p ds \right) \left(\int_{\frac{1}{10}}^s e^{\mu t} t^{pb} dt \right) \\ & \leq \frac{C_{l_\mu, p, b}}{\mu^{p(1+\gamma)}} \int_0^\infty f^p s^{pb} ds. \quad \text{The proof of (25) is complete.} \end{aligned}$$

The proof of (26) is similar. \square

Proof of Remark 1.8: We only have to show the "only if". Without loss of generality, we only consider the Schauder theory, and assume $m_0 = 1$, $\gamma = b = 1$, $\vec{\beta} = 0$, $\sigma_1 = \sigma_2 = Id$. For any $\phi_0 \in Ker B_{P^0}$, let $f = \frac{\phi_0}{t}$. When $P = \frac{\partial}{\partial t} - B$ (translation-invariant on the end) and $t \geq 10$, the general solution to $Pu = \frac{\phi_0}{t}$ is $(\log t + C)\phi_0 \notin \widehat{C}_{0,0}^{1,\alpha}$. Hence any extension \tilde{f} of f to the whole N is not in $Range P|_{\widehat{C}_{0,0}^{1,\alpha}}$.

On the other hand, let $f_k = \frac{\phi_0}{t^{1+\frac{1}{k}}}$, Theorem 3.2 implies each f_k admits an extension $\tilde{f}_k \in Range P|_{\widehat{C}_{0,0}^{1,\alpha}}$, and \tilde{f}_k tends to $\tilde{f} \in C_{0,1}^\alpha$ which extends f . Then $Range P|_{\widehat{C}_{0,0}^{1,\alpha}}$ is not closed in $C_{0,1}^\alpha$. \square

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